Thermodynamically Consistent Constitutive Equations for Nonisothermal Large-Strain, Elastoplastic, Creep Behavior

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The paper is concerned with the development of constitutive relations for large nonisothermal elastic-viscoplastic deformations for metals. The kinematics of elastic-plastic deformation, valid for finite strains and rotations, is presented. The resulting elastic-plastic uncoupled equations for the deformation rate combined with use of the incremental elasticity law permits a precise and purely deductive development of elastic-viscoplastic theory. It is shown that a phenomenological thermodynamic theory in which the elastic deformation and the temperature are state variables, including few internal variables, can be utilized to construct elastic-viscoplastic constitutive equations appropriate for metals. The limiting case of inviscid plasticity is examined.

Nomenclature

da	= element of area
ds	= material line element
d_{rs}	= deformation rate
$E_{\alpha\beta}$	= strain rate
F	= yield function
f^A_α, F^r_α	= deformation gradients
g_r, G^A	= base vectors
g^{rs}, G^{AB}	= metric tensors
J	= absolute determinant of the deformation gradient
$k, \alpha^A_C, A^{AB}_{CD}$	= internal variables
$\boldsymbol{n}$	= normal to the surface
$\boldsymbol{P}$	= force
q	= specific applied heat
s	= entropy
T	= temperature
t	= fraction sector
t	= time
u	= specific internal energy
V	= volume
v^r	= velocity
w	= specific mechanical work
W_{rs}	= spin tensor
x^i	= inertial coordinate system
x^α	= material coordinate system
x^A	= convected coordinate system
ρ	= density
$\boldsymbol{\sigma}$	= Cauchy stress tensor
$\boldsymbol{\tau}$	= Kirchhoff stress tensor
ϕ	= specific free energy
$\overset{\nabla}{\sigma}$	= Jauman stress rate
$\dot{\sigma}$	= time derivative

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Introduction

THE prediction of inelastic behavior of metallic materials at elevated temperature has increased in importance in recent years. Many important engineering applications involve the use of metals subjected to cyclic thermomechanical loads, e.g., hot section components of turbine engines, nuclear reactor components, etc. These materials exhibit substantial complexity in their thermomechanical constitution. In fact, so complex is their material response that it could be argued that without useful a priori information, experimental characterization is futile. It is, therefore, important to be able to model accurately the nonelastic behavior of metals under cyclic mechanical and thermal loading at temperature levels for which creep and recovery introduce significant response phenomena.

Under this kind of severe loading conditions, the real world of structural behavior is highly nonlinear due to the combined action of geometrical and physical nonlinearities. On one side, finite deformation (in a stressed structure) introduce nonlinear geometric effects. On the other side, physical nonlinearities arise even in small strain regimes, whereby inelastic phenomena play a particularly important role. From a theoretical standpoint nonlinear constitutive equations should be applied only in connection with nonlinear deformation measures. However, in engineering practice, the two sources of nonlinearities are separated for practical reasons, yielding at one end of the spectrum large displacement and large rotation problems and on the other end inelastic analysis in the presence of small strain.

Constitutive models for small strain in engineering literature may generally be grouped into three categories: classical plasticity, nonlinear viscoelasticity, and theories based on microstructural phenomena. Each group can be further separated into "unified" and "uncoupled" theories, where the two differ in their approach to the treatment of rate-independent and rate-dependent inelastic deformation. The uncoupled theories decompose the inelastic strain rate into a time-independent plastic strain rate and a time-dependent creep rate with independent constitutive relations describing plastic and creep behavior. Such uncoupling of the strain components provides for simpler theories to be developed, but precludes any creep/plasticity interaction. Recognizing that cyclic plasticity, creep, and recovery are not independent phenomena but rather are very interdependent, a number of "unified" models for inherently time-dependent nonelastic deformation have been developed recently.

Classical incremental plasticity theories are macrophenomenological because they base the derivation of state variables purely on experimental results without direct reference to the microstructure of the material. Most incremental plasticity theories have four major components: 1) a stress-elastic strain relation, 2) a yield function describing the onset of plastic deformation, 3) a hardening rule that prescribes the strain hardening of the material and the modification of the yield surface during plastic flow, and 4) a flow rule that defines the components of strain that is plastic or nonrecoverable. Research in this area is voluminous. For example, Zienkiewicz and Cormeau[1] developed a rate-dependent unified theory that allows for nonassociative plasticity and strain softening, but does not model the Bauschinger effect or temperature dependence. Extensions of classical plasticity to model both rate and temperature effects were presented recently by Allen and Haisler,[2] Haisler and Cronenworth,[3] and Yamada and Sakurai.[4]

In the nonlinear viscoelastic approach, the constitutive relation is expressed as a single integral or convoluted form. This type of constitutive model employs the thermodynamic laws along with physical constraints to complete the formulation. A detailed review of several existing theories is presented by Walker.[5] Walker's theory is based on a unified viscoplastic integral developed by modifying the constitutive relations for a linear three-parameter viscoelastic solid. The theory contains clearly defined material parameters, a rate-dependent equilibrium stress, and a proposed multiaxial model. An important shortcoming of Walker's theory is its failure to model transient temperature conditions. Many other nonlinear viscoelastic theories have been proposed, including those by Cernocky and Krempl,[6] Valanis,[7] and Chabache.[8]

The microphenomenological theories attempt to represent the response of polycrystalline materials in terms of various micromechanisms of deformation and failure. Various dislocation theories have been developed to predict plastic deformation in terms of dislocation interaction, slip, glide, density, etc. Most of the material models developed to date depend primarily on the number of state variables used and their growth or evolutionary laws. Many of the recent "unified" microphenomenological theories have been discussed and evaluated by Walker[9] and Chan et al.[10]

One example of a microphysically based constitutive law is an elastic-viscoplastic theory based on two internal state variables as proposed by Bodner et al.[11] These authors demonstrate the ability of the constitutive equations to represent the principal features of cyclic loading behavior, including softening upon stress reversal, cyclic hardening or softening, cyclic saturation, cyclic relaxation, and cyclic creep. One limitation of the formulation though is that the computed stress-strain curves are independent of the strain amplitude and therefore too "flat" or "square."

Miller[12] has reported research on the modeling of cyclic plasticity with "unified" constitutive equations. He also recognizes the shortcomings of many theories in predicting hysteresis loops that are oversquare in comparison to observed experimental behavior. Improvement is accomplished by making the kinematic work-hardening coefficient depend on the back stress and the sign of the nonelastic strain term. Theories that are similar in format to Miller's have been proposed by Krieg et al.[13] and Hart.[14] The models use two internal state variables to reflect current microstructure state and are based upon models for dislocation processes in pure metals. All of these constitutive theories were formulated without the use of a yield criterion. Since these models do not contain a completely elastic regime, the function that describes the inelastic strain rate should be such that the inelastic strain rate is very small for low stress levels. Theories with a yield function and a full elastic regime have been developed for the case of isotropic hardening by Robinson[15] and Lee and Zavrel[16] for both isotropic and directional hardening.

As previously noted, the quantities utilized in the small strain theory of viscoplasticity (stress, strain, stress rate, and strain rate) are defined only within the assumption of "small strains." Yet the precise definition of what constitutes "small strain" is always left unstated. Whether or not the stresses for a given case are "small" cannot be determined a priori by geometric considerations. In general, one cannot know in advance whether, for a given loading of a material, the "small-strain" assumption (always left undefined) will hold or not. The question of whether the small-strain approximations are valid is always avoided in the "small-strain" literature. Furthermore, as Hill[17] points out, the really typical plastic problems involve changes in geometry that cannot be disregarded. In many cases, for example, it is sufficient to take into account finite plastic strains and small elastic strains or vice versa. From the theoretical viewpoint, it is desirable in all cases to have a theory that intrinsically allows for both the elastic and plastic strains to be large. Such a theory, of course, must reduce to the earlier mentioned special cases, as limiting cases. Furthermore, such theories provide a check for those obtained by generalizing small-strain theories.

The mathematical theories of deformation and flow of matter deal essentially with the gross properties of a medium. Heat and mechanical work are considered as additional causes for a change of the state of the medium. The resulting phenomena in any particular material are not unrelated. Therefore, a thermodynamical treatment of the foundation of the theory of flow and deformation is appropriate and, indeed, the obvious approach. Two very different main approaches to a thermodynamic theory of a continuum can be identified. These differ from each other in the fundamental postulates upon which the theories are based. An essential controversy (a good survey of this controversy is given in Ref. 18) can be traced through the whole discussion of the thermodynamic aspects of continuum mechanics. None of these approaches is concerned with the atomic structure of the material. Therefore, they represent purely phenomenological approximations. Both theories are characterized by the same fundamental requirement that the results should be obtained without having recourse to statical or kinetic methods.

Within each of these approaches, there are two distinct methods of describing history and dissipative effects: the functional theory[19] in which all dependent variables are assumed to depend on the entire history of the independent variables and the internal variable approach[20] wherein history dependence is postulated to appear implicitly in a set of internal variables. For experimental as well as analytical reasons,[21,22] the use of internal variables in modeling inelastic solids is gaining widespread usage in current research. The main differences among the various modern theories lie in the choice of these internal variables.

Therefore, the predictive value of an elastic-viscoplastic material model for nonisothermal, large-deformation analyses depends on three basic elements: 1) the nonlinear kinematic description of the elastic-plastic deformation, 2) thermodynamic considerations, and 3) the choice of external and internal thermodynamic variables. The objective of this paper is to examine each of these elements, illustrate their interaction, and extend these considerations to model the large, nonisothermal, elastic-viscoplastic deformation behavior of metals.

Moreover, the paper deals with the phenomenological theory of elastic-viscoplastic bodies. The process inside the lattice and at the border of the crystal grains is taken as the physical background, without considering its connection to the macroscopic behavior of the material at the present.

Kinematic and Fundamental Considerations

Consider body of volume V that occupies a finite region of Euclidean space. When subjected to prescribed body

forces, surface tractions, surface temperature, and surface velocities, the body undergoes motion characterized by $x^i = \chi^i(X^\alpha, t)$. The material particles of the body are identified by coordinates X^α, which are referred to as material coordinates. The relation of the material particles to the material coordinates X^α does not change in time. The places in space that the particles occupy during the motion are identified by the coordinates x^i. Functions χ^i describe the motion of the particles X^α through space. The place occupied by the body at $t=0$ is taken as the initial configuration. In this configuration the body is assumed to be strain-free, but not necessarily stress-free.

A third coordinate system is defined by the material coordinates as they deform with the body. This system will be denoted by X^A, which are referred to as convected coordinates. The current configuration of the body with spatial coordinates x^i and convected coordinates X^A and the initial configuration of the body with material coordinates X^α will be employed in what follows. For the spatial coordinates x^i, the covariant base vectors $\boldsymbol{g}_r$, the contravariant base vector $\boldsymbol{g}^r$, the metric g_{rs}, and its dual g^{rs} are used. Similarly, for the convected coordinates X^A, the covariant base vectors $\boldsymbol{G}_A$, the contravariant base vectors $\boldsymbol{G}^A$, the metric tensor G_{AB}, and its dual G^{AB} are used. With regard to the initial configuration, the covariant base vectors $\boldsymbol{G}\alpha$, the contravariant base vectors $\boldsymbol{G}^\alpha$, the metric tensor $G_{\alpha\beta}$, and it dual $G^{\alpha\beta}$ are used for the material coordinates X^α.

For a second-order tensor $\boldsymbol{A}$ with components A^{rs} in the spatial coordinates and components A^{AB} in the convected coordinates, the following is true:

$$\boldsymbol{A} = A^{rs}\boldsymbol{g}_r\boldsymbol{g}_s = A^{AB}\boldsymbol{G}_A\boldsymbol{G}_B \tag{1}$$

The two sets of components are related to each other through

$$A^{rs} = x^r_{,A}x^s_{,B}A^{AB} \tag{2}$$

where $X^r_{,A}$ denotes the partial derivative $\partial\chi^r(X^B,t)/\partial X^A$.

For the motion, characterized by $\chi^r(X^A,t) = \chi^r(X^\alpha,t)$, we have

$$\boldsymbol{G}_A = x^r_{,A}\boldsymbol{g}_r \qquad G_{AB} + X^r_{,A}x^s_{,B}g_{rs} \tag{3}$$

From Eq. (3), it is seen that $\dot{G}_{AB} = 0$, where the dot denotes time material derivative. The tensor transformation equations (1) and (2) will be used extensively in what follows.

A material line element $d\boldsymbol{s} = dX^\alpha\boldsymbol{G}^\alpha$ in the initial configuration when subjected to motion $\chi^r(X^\alpha,t)$ is deformed into $d\boldsymbol{s} = dx^r\boldsymbol{g}_r$ in the current configuration. The line element dx^r is related to the line element dX^α through the deformation gradient F^r_α by $dx^r = F^r_\alpha dX^\alpha$ where

$$F^r_\alpha = \frac{\partial x^r}{\partial x^\alpha}(x^\beta, t) \tag{4}$$

The mapping defined by the deformation gradient $\boldsymbol{F} = F^r_\alpha\boldsymbol{g}_r\boldsymbol{G}^\alpha$ allows one to shift quantities from the current configuration to equivalent, but alternate, quantities in the initial configuration. For example, the right Caushy-Green tensor $\boldsymbol{C} = C_{\alpha\beta}\boldsymbol{G}^\alpha\boldsymbol{G}^\beta$ and the Green-St. Venant strain tensor $\boldsymbol{E} = E_{\alpha\beta}\boldsymbol{G}^\alpha\boldsymbol{G}^\beta$, in the initial configuration are

$$\begin{aligned} d\boldsymbol{s} = d\boldsymbol{S} &= g_{rs}dx^r dx^s - G_{\alpha\beta}dX^\alpha dX^\beta \\ &= (g_{rs}F^r_\alpha F^s_\beta - G_{\alpha\beta})dX^\alpha dX^\beta \\ &= (C_{\alpha\beta} - G_{\alpha\beta})dX^\alpha dX^\beta = 2E_{\alpha\beta}dX^\alpha dX^\beta \end{aligned} \tag{5}$$

The components of the deformation gradient, which relate a deformed line element dX^A in the convected coordinates to the undeformed line element dX^α in the initial configuration, are given by f^A_α,

$$F^r_\alpha = x^r_{,A}f^A_\alpha \tag{6}$$

Equation (6) places in a single expression the easily confused but distinct ideas of the transformation of tensor components under a change of coordinates and a shift between the current configuration and the initial configuration as a setting for the governing equations. Truesdell and Toupin[23] and Truesdell and Noll[24] emphasizes the current configuration with the spatial coordinates and an initial configuration with material coordinates. As a result, the deformation gradient plays a prominent role in their work. Only in isolated spots do they mention convected coordinates and, then, as indirectly as possible. On the other hand, Green and Adkins[25] and Sedov[26] rely heavily on convected coordinates. Our intention here is only to tie the two together for the purpose of discussing elementary assumptions. Recently, Mendelssohn and Baruch[27] review this same point as well as additional material relevant to sound numerical formulation of finite deformation problems.

The velocity $\boldsymbol{v} = v^r\boldsymbol{g}_r$ of a particle X^α is defined by

$$v^r \equiv \frac{\partial x^r}{\partial t}(X^\alpha, t) \tag{7}$$

From the spatial gradient of the velocity, the deformation rate

$$\boldsymbol{d} = d_{rs}\boldsymbol{g}^r\boldsymbol{g}^s = d_{AB}\boldsymbol{G}^A\boldsymbol{G}^B \tag{8}$$

is defined as

$$d_{rs} \equiv \tfrac{1}{2}(V_{r,s} + V_{s,r}) \tag{9}$$

The spin

$$\boldsymbol{W} = W_{rs}\boldsymbol{g}^r\boldsymbol{g}^s = W^{AB}\boldsymbol{G}_A\boldsymbol{G}_B \tag{10}$$

is defined as

$$W_{rs} \equiv \tfrac{1}{2}(V_{r,s} - V_{s,r}) \tag{11}$$

In the initial configuration, the Green-St. Venant strain rate is the shifted deformation rate,

$$\dot{E}_{\alpha\beta} = F^r_\alpha F^s_\beta d_{rs} = f^A_\alpha f^B_\beta d_{AB} \tag{12}$$

Basic to most of the postulated models of large elastic-plastic deformation behavior is the additive decomposition of d_{rs} and $E_{\alpha\beta}$ into elastic and plastic parts,[28]

$$d_{rs} = \overset{E}{d}_{rs} + \overset{P}{d}_{rs}, \quad \dot{E}_{\alpha\beta} = \overset{E}{\dot{E}}_{\alpha\beta} + \overset{P}{\dot{E}}_{\alpha\beta} \tag{13}$$

The validity of this additive decomposition in the case of finite elastic-plastic strains has been questioned by Lee and his associates.[29-32] Lee's[29] approach is based on the total purely elastic unloading from the current state to an intermediate unstressed plasticity deformed configuration, without any reverse or other kind of plastic flow. The major point in his theory is to decouple the total elastically induced distortion and measure it from a relaxed unstressed state, which is only plastically deformed from the initial to the intermediate configuration. Accordingly, the deformation gradient $\boldsymbol{F}$ is decomposed in the form

$$\boldsymbol{F} = \overset{E}{\boldsymbol{F}}\,\overset{P}{\boldsymbol{F}} \tag{14}$$

where $\overset{P}{\boldsymbol{F}}$ transforms a line element from the initial configuration to the intermediate configuration and $\overset{E}{\boldsymbol{F}}$ from the latter

to the current configuration. The intermediate configuration is chosen in such a way so that $\overset{P}{F}$ is unaffected by the presence of rigid-body motion. The deformation rate tensors d_{sr} and d_{rs} are then defined. After some manipulations, Lee shows the following relation:

$$d_{rs} = \overset{E}{d}^{rs} + \overset{E}{F}_{rk} d_{kl} \overset{E}{F}^{-1}_{rs} \Big|_s + \overset{E}{F}_{rk} \overset{P}{W}_{rs} \overset{E}{F}^{-1}_{ls} \Big|_s \tag{15}$$

where the subscripts s denote the symmetric parts. Generalization of Lee's theory for anisotropic elasticity was given by Mandel.[33]

Lee's theory is based on the assumption that the elastic law does not change with the history of deformation and, hence, a total elastic unloading can take place. However, it has been shown[34] that after a fair amount of plastic flow has taken place, reverse plastic deformation will result soon upon unloading, even for small strains. Therefore, a total elastic unloading cannot have any physical significance. In view of this, the theory of Lee appears as a special case of the theory of Green and Naghdi.[35] Although not as general as the theory of Green and Naghdi, Lee's theory has the advantage of being more easily fitted with the physical property of invariance of elasticity with respect to plastic deformation. In particular, Mandel[33] has pointed out that the Green-Naghdi theory is not convenient if one wants to include anisotropic elasticity effects. All this can be avoided by the use of the convected coordinates, as proposed by Sedov[26] and Lehmann.[36] The formulation presented herein will follow the work of Lehmann.

All quantities from here on will be related to the metric of the coordinate system X^A in the deformed state. Hence,

$$\begin{aligned} f^{\alpha}_{c} &= G^{\alpha\beta} G_{BC} \\ (f^{-1})^{A}_{\gamma} &= G^{AB} G_{\beta\gamma} \end{aligned} \tag{16}$$

and the deformation rate is

$$\begin{aligned} d^{A}_{r} &= \tfrac{1}{2} G^{AB} \dot{G}_{\beta\gamma} = -\tfrac{1}{2} G_{CB} \dot{G}^{\beta\alpha} \\ &= \tfrac{1}{2} (f^{-1})^{A}_{\beta} (\dot{f})^{\beta}_{\cdot\gamma} = -\tfrac{1}{2} (\dot{f}^{-1})^{A}_{\cdot B} f^{B}_{\gamma} \end{aligned} \tag{17}$$

The deformation gradient may be split into its elastic and its plastic components in the following manner:

$$\begin{aligned} f^{\alpha}_{c} &= \underbrace{G^{\alpha\beta}\, G_{\beta\tau}}_{\overset{P}{F}^{\alpha}_{\cdot\Gamma}} \underbrace{G^{\Gamma D} G_{DC}}_{\overset{E}{f}^{\Gamma}_{c}} \\ (f^{-1})^{A}_{\gamma} &= \underbrace{G^{AB}\, G_{B\tau}}_{\overset{E}{(f^{-1})}^{A}_{\Gamma}} \underbrace{G^{\Gamma\delta}\, G_{\delta\gamma}}_{\overset{P}{(f^{-1})}^{\Gamma}_{\cdot\gamma}} \end{aligned} \tag{18}$$

The use of capital greek subscripts and superscripts ($G_{\beta\Gamma}$) denotes parameters belonging to a fictitious intermediate state, which is in general incompatible. The circumstance of the noncontinuous configuration in the unstressed state has been observed by Sedov,[26] who points out that convected coordinates, as used herein, become non-Euclidean in this configuration.

This multiplicative splitting of the metric change in the convected coordinates leads to an additive splitting of the deformation rate according to

$$\begin{aligned} d^{A}_{c} &= \operatorname{sym}\tfrac{1}{2} \{ \overset{E}{(f^{-1})}^{A}_{\Gamma} \overset{E}{(\dot{f})}^{\Gamma}_{\cdot C} \} + \operatorname{sym}\tfrac{1}{2} \{ \overset{P}{(f^{-1})}^{A}_{\delta} (\dot{f})^{\delta}_{\cdot\Gamma} \overset{E}{f}^{\Gamma}_{c} \} \\ &= \operatorname{sym}\tfrac{1}{2} \{ \overset{E}{(\dot{f}^{-1})}^{A}_{\cdot\Gamma} \overset{E}{f}^{\Gamma}_{C} \} - \operatorname{sym}\tfrac{1}{2} \{ \overset{E}{(f^{-1})}^{A}_{\Gamma} \overset{P}{(\dot{f}^{-1})}^{\Gamma}_{\cdot\delta} f^{\delta}_{C} \} \\ &= \overset{E}{d}^{A}_{C} + \overset{P}{d}^{A}_{C} \end{aligned} \tag{19}$$

In the current configuration of the body V, consider an element of area da on the surface of $\mathcal{S}$ with an outward normal $\boldsymbol{n} = n_r \boldsymbol{g}^r = n_A \boldsymbol{G}^A$. If the force $d\boldsymbol{P} = dP^r \boldsymbol{g}_r = dP^A \boldsymbol{G}_A$ is acting on this element, the fraction vector is $\boldsymbol{t} = d\boldsymbol{P}/da$. The Caushy stress,

$$\boldsymbol{\sigma} = \sigma^{rs} \boldsymbol{g}_r \boldsymbol{g}_s = \sigma^{AB} \boldsymbol{G}_A \boldsymbol{G}_B \tag{20}$$

defined, such that $\boldsymbol{t} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$, which in component form (in terms of spatial coordinates) is $t^r = \sigma^{rs} n_s$. In the convected coordinates, it is $t^A = \sigma^{AB} n_B$.

It is convenient to work with the Kirchhoff stress tensor τ in the current configuration, obtained from the Cauchy stress by scaling

$$\tau^{A}_{B} = \frac{\rho_0}{\rho} \sigma^{A}_{B} J \sigma^{A}_{B} \tag{21}$$

where ρ denotes the current mass density, ρ_0 the mass density in the initial state, and J the absolute determinant of the deformation gradient at the current configuration.

The time derivative of a tensor such as stress, which is associated with the current configuration, admits infinitely many definitions, depending upon the coordinate system employed in the computation of such time derivatives. For a correct large-strain, large-rotation elastic-plastic model, the notion of "invariant stress fluxes" and "objectivity" must be introduced. A good treatment requires more space than is available here.[23] The corotational stress rate, also referred to as Jauman stress rate, will suffice for the purpose of the present discussion. Hence, in convected coordinates,

$$\begin{aligned} \overset{\nabla}{\sigma}{}^{A}_{B} &= \dot{\sigma}^{A}_{\cdot B} + d^{A}_{C} \sigma^{C}_{B} - d^{C}_{B} \sigma^{A}_{C} \\ \overset{\nabla}{\tau}{}^{A}_{B} &= \dot{\tau}^{A}_{B} + d^{A}_{C} \tau^{C}_{B} - d^{C}_{B} \tau^{A}_{C} \end{aligned} \tag{22}$$

From Eq. (21), the following relations between the various rates of Kirchoff stress and Cauchy stress are obtained:

$$\begin{aligned} \dot{\tau}^{A}_{B} &= \frac{\rho_0}{\rho} \dot{\sigma}^{A}_{B} + J d^{C}_{C} \sigma^{A}_{B} \\ \overset{\nabla}{\tau}{}^{A}_{B} &= \frac{\rho_0}{\rho} \overset{\nabla}{\sigma}{}^{A}_{B} + J d^{C}_{C} \sigma^{A}_{B} \end{aligned} \tag{23}$$

If a rate constitutive law is postulated between $\dot{\boldsymbol{\sigma}}$ and $\boldsymbol{d}$ in finite inelasticity theories, then a potential does not exist, which is necessary in the variational or thermodynamics-based formulation of the problem. The basic difficulty lies with the d^{c}_{c} term. This is remedied by postulating a constitutive law between $\dot{\boldsymbol{\tau}}$ and $\boldsymbol{d}$.

The Elastic Deformations

The present study is concerned with the structure of the constitutive relation of an elastic-viscoplastic (elastic-plastic) medium. The term elastic-viscoplastic means that the viscosity does not intervene in the elastic domain whose boundary, in particular, is well defined at every stage of the deforma-

tion. For simplicity, we assume further that the thermoelastic behavior of the body is isotropic and unaffected by inelastic deformation in the sense that the material constants characterizing the thermoelastic behavior are independent of inelastic deformation. Thus, we can obtain a unique relation between the elastic deformations represented by $\overset{E}{f}{}^A_C$, the Kirchhoff stresses τ^A_C, and the temperature T,[35,36]

$$\overset{E}{f}{}^A_C = \overset{E}{f}{}^A_C(\tau^A_C, T); \quad \tau^A_C = \tau^A_C(\overset{E}{f}{}^B_D, T); \quad T = T(\tau^A_C, \overset{E}{f}{}^B_D) \tag{24}$$

This function may be transformed into an incremental relation by differentiation with respect to time. This leads to a general expression of the form

$$\overset{E}{d}{}^A_C = \overset{E}{d}{}^A_C\{\tau^A_C, \overset{\nabla}{\tau}{}^A_C, T, \dot{T}, G_{AC}, d^A_C\} \tag{25}$$

From Eq. (24) we see that the total deformation rate enters the incremental form of the thermoelastic stress-strain relations. Therefore, the thermoelastic deformation is not independent of the inelastic deformation occurring at the same time. This follows from the fact that in the integrated form of the thermoelastic stress-strain relations Eq. (24), the stresses and the strains are referred to the deformed state of the body.

In view of the present discussion and the discussion in the previous section, the hyperelastic behavior described by Eqs. (24) and (25) will be replaced by a hypoelastic law. The hypoelastic law is a path-dependent material law, since it cannot be expressed in terms of an initial and a final state; it depends on the path connecting these states. Otherwise, if we did not make such a change, it would be necessary to retain the finite deformation measure in the constitutive law. For small elastic strains, there is practically no difference between hypoelastic and hyperelastic laws, as shown, for example, by Lehmann.[36]

The above could be illustrated by the following example. From the frequently used elastic stress-strain relation,

$$\epsilon^A_C = \tfrac{1}{2}\left\{\delta^A_C - (\overset{E}{f}{}^{-1})^A_C\right\} = \frac{1}{2G}\left\{\tau^A_C - \frac{\nu}{1+\nu}\tau^B_B\delta^A_C\right\} + \alpha(T - T_0)\delta^A_C \tag{26}$$

we get

$$\overset{E}{d}{}^A_C = \frac{1}{2G}\left\{\mathrm{sym}\left[\overset{E}{f}{}^D_C(\dot{\tau})^A_{.D}\right] - \frac{\nu}{1+\nu}\overset{E}{f}{}^A_C(\dot{\tau}^B_B)\right\} + \alpha\dot{T}\overset{E}{f}{}^A_C \tag{27}$$

which may be replaced by

$$\overset{E}{d}{}^A_C = \frac{1}{2G}\left\{\overset{\nabla}{\tau}{}^A_C - \frac{\nu}{1+\nu}\overset{\nabla}{\tau}{}^B_B\delta^A_C\right\} + \alpha\dot{T}\delta^A_C + \alpha\dot{T}\delta^A_C \tag{28}$$

We assume that inelastic deformation occurs if and only if

$$F(\tau^A_C, T, k, \ldots, \alpha^A_C, \ldots, A^{AB}_{CD}, \ldots) = 0 \tag{29a}$$

$$\frac{\partial F}{\partial \tau^A_C}\overset{\nabla}{\tau}{}^A_C + \frac{\partial F}{\partial T}\dot{T} > 0 \tag{29b}$$

or, for elastic-plastic material,

$$F(\tau^A_C, T, k, \ldots, \alpha^A_C, \ldots, A^{AB}_{CD}, \ldots) > 0 \tag{30}$$

for an elastic-viscoplastic material. The function F represents the yield condition that bounds the domain of pure thermoelastic behavior in the ten-dimensional space of stress and temperature. The inequality given by Eq. (29b) is the loading condition. The actual form of the yield condition for a given material is determined by a set of so-called internal parameters, which are scalars and/or tensors of even order. The current values of the internal parameters depend on the initial state of the material and the history of the thermomechanical process.

Thermodynamic Processes

In the treatment of elastic-plastic or elastic-viscoplastic deformations, we have to distinguish between the description as a thermomechanical process and the corresponding one by means of thermodynamic state equations. It is sometimes assumed that, in the case of processes which proceed through nonequilibrium states, it is fundamentally necessary to start with a description of the process.[19,24,37] Alternatively, it has been proposed that one might assume local equilibrium for the elements of a body and therefore describe the state of the elements, in general, by state equations.[38-40] The consequences of adopting these two approaches become particularly clear when considering the influence of entropy. In the description of the process, entropy is a derived quantity and in principle we can proceed without introducing it. In the description by state equations, it is, on the contrary, a necessary state value that, at least in principle, can be immediately determined. When restricting ourselves to homogeneous, quasistatical thermomechanical processes, the description by state equations can be reviewed as equivalent to that by processes.[37,41] The controversial issues will, thus, not be discussed further.

Restricting ourselves to elementary processes, we need not analyze whether the applied heat arises from heat conduction or from heat sources. For the same reason, it is not necessary, in our case, to introduce the temperature gradient in addition to the temperature or the body forces in addition to the stresses.

The first law states, under our simplifying assumptions, that the rate of the specific internal energy $\dot{u}$ is the sum of the rates of the specific mechanical work $\dot{w}$ and the specific applied heat $\dot{q}$,

$$\dot{u} = \dot{w} + \dot{q} \tag{31}$$

The rate of mechanical work is given by

$$\dot{W} = \frac{1}{\rho_0}\tau^A_C d^C_A \tag{32}$$

and may be split into an elastic and an inelastic part according to Eq. (19),

$$\dot{W} = \frac{1}{\rho_0}\overset{E}{\tau}{}^A_C d^C_A + \frac{1}{\rho_0}\overset{P}{\tau}{}^A_C d^C_A = \overset{E}{\dot{W}} + \overset{I}{\dot{W}} \tag{33}$$

The rate of inelastic work must also be split into a part $\overset{D}{\dot{W}}$, which is dissipated at once, and into another part $\overset{S}{\dot{W}}$, which represents changes in the internal state. Thus,

$$\overset{I}{\dot{W}} = \frac{1}{\rho_0}\overset{P}{\tau}{}^A_C d^C_A = \overset{S}{\dot{W}} + \overset{D}{\dot{W}} \tag{34}$$

Only $\overset{D}{\dot{W}}$ enters the entropy production

$$T\dot{S} = \dot{q} + \overset{D}{\dot{W}} \tag{35}$$

The second law of thermodynamics requires

$$\overset{D}{\dot{W}} \geq 0 \tag{36}$$

We use as thermodynamic state variables the elastic strain, represented by $\overset{E}{f}{}^A_C$, the absolute temperature T, and a number of other internal state variables $(k,\ldots,\alpha^A_C,\ldots,A^{AB}_{CD},\ldots)$ that may be scalars and tensors of even order. The choice of $\overset{E}{f}{}^A_C$ and T as state variables is based on the fact that in pure thermoelastic deformations, both quantities form a suitable set of thermodynamic state variables. The plastic strain and the total strain are unsuitable as state variables because, in general, they do not uniquely define the state of the material. A conflicting point of view has been expressed in Refs. 42–44. The remaining state variables are added for the sake of the description of the changes of the internal structure of the material.

The specific free energy (Helmholtz function) ϕ given by

$$\phi = u - Ts \tag{37}$$

must be a unique function of the thermodynamic state variables

$$\phi = \overset{e}{\phi}(f^A_C, T, k, \alpha^A_C, \ldots, A^{AB}_{CD}, \ldots) \tag{38}$$

Since the elastic part of the deformation, according to our assumptions, does not depend on the plastic deformation, we may divide the free energy into two different components, as

$$\phi = \overset{E}{\phi}(\overset{E}{f}{}^A_C, T) + \overset{S}{\phi}(T,\ k,\ldots,\alpha^A_C,\ldots,A^{AB}_{CD},\ldots) \tag{39}$$

where the first component $\overset{E}{\phi}$ refers to the elastic deformation and the second $\overset{S}{\phi}$ to the changes of the internal state.

From Eqs. (31), (33–35), and (37) we derive

$$\dot{\phi} = -s\dot{T} + \overset{E}{\dot{W}} + \overset{S}{\dot{W}} \tag{40}$$

Also, we obtain from Eq. (39)

$$\dot{\phi} = \frac{\partial \overset{E}{\phi}}{\partial \overset{E}{f}{}^A_C} \overset{E}{\overset{\nabla}{f}}{}^A_C + \frac{\partial (\overset{E}{\phi} + \overset{S}{\phi})}{\partial T}\dot{T} + \frac{\partial \overset{S}{\phi}}{\partial k}\dot{k} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial \alpha^A_C}\overset{\nabla}{\alpha}{}^A_C + \ldots + \frac{\partial \overset{S}{\phi}}{\partial A^{AB}_{CD}}\overset{\nabla}{A}{}^{AB}_{CD} + \ldots \tag{41}$$

By comparison of Eqs. (40) and (41), we may conclude that

$$S = -\frac{\partial (\overset{E}{\phi} + \overset{S}{\phi})}{\partial T}$$

$$\overset{S}{\dot{W}} = \frac{\partial \overset{S}{\phi}}{\partial k}\dot{k} + \ldots + \frac{\partial \overset{S}{\phi}}{\partial \alpha^A_C}\overset{\nabla}{\alpha}{}^A_C + \ldots + \frac{\partial \overset{S}{\phi}}{\partial A^{AB}_{CD}}\overset{\nabla}{A}{}^{AB}_{CD}\ldots$$

$$\tau^A_C = \rho_0 \overset{E}{f}{}^B_C \frac{\partial \overset{E}{\phi}}{\partial \overset{E}{f}{}^A_B} \tag{42}$$

For irreversible processes, this scheme of description has to be completed by some statements about the dependence of entropy production on the thermomechanical process. Under our assumption, we need deal only with entropy production by dissipated mechanical work, in connection with inelastic deformation. Thus, we assume, in general

$$\overset{D}{\dot{W}} = C^{AB}_{CD}\tau^C_A \overset{I}{d}{}^D_B > 0 \tag{43}$$

where

$$C^{AB}_{CD} = \overset{E}{C}{}^{AB}_{CD}(\overset{E}{f}{}^A_C, T, k, \ldots, \alpha^A_C, \ldots, A^{AB}_{CD}) \tag{44}$$

Equations (42) and (43) are the governing equations for nonisothermal, elastic-inelastic elementary processes. The specific free energy ϕ, which determines the nondissipated work of the thermomechanical process and the quantity C^{AB}_{CD}, which governs the entropy production, must be specified according to the material behavior.

Elastic-Viscoplastic Model

Thermomechanical processes in elastic-viscoplastic bodies cannot be considered as a sequence of equilibrium states, even in the case of the elementary processes considered here. Elastic-viscoplastic deformations are associated with nonequilibrium states. One consequence of this fact is that we may get a continuation of a process without any change in the independent process variables. This occurs, for example, in the case of creep with constant stress and temperature or in the case of an adiabatic stress relaxation under constant strain. In such cases, the body moves from a nonequilibrium state to an equilibrium state.

In order to establish the constitutive relations for elastic-viscoplastic bodies, which in the limiting case becoming elastic-inviscidly plastic, we adopt the usual assumption that the stresses, which produce the inelastic deformation, may be expressed as the sum of the so-called athermal or inviscid stresses, $\bar{\tau}^A_C$ and the viscous overstresses $\overset{*}{\tau}{}^A_C$

$$\tau^A_C = \bar{\tau}^A_C = \overset{*}{\tau}{}^A_C + (\tau^A_C - \bar{\tau}^A_C) \tag{45}$$

This assumption by no means detracts from the "unified" concept. The rate-independent limit of viscoplastic constitutive relation was recently discussed by Travnicek and Kratochvil.[45] Hence, the total work rate can be partitioned in the following way:

$$\dot{W} = \overset{E}{\dot{W}} + \underbrace{\overset{P}{\dot{W}} + \overset{V}{\dot{W}}}_{\overset{I}{\dot{W}}} = \frac{1}{\rho_0}\tau^A_C \overset{E}{d}{}^C_A + \frac{1}{\rho_0}\bar{\tau}^A_C \overset{P}{d}{}^C_A + \frac{1}{\rho_0}\overset{*}{\tau}{}^A_C \overset{P}{d}{}^C_A \tag{46}$$

The viscous part of the work is completely dissipated. Thus, we may write

$$\overset{V}{\dot{W}} = \overset{D_v}{\dot{W}} \tag{47}$$

Regarding the plastic work, we have already stated that one part is used for changing the internal state and only the remaining part can be considered to be dissipated. Therefore, we must write

$$\overset{P}{\dot{W}} = \overset{S}{\dot{W}} + \overset{D_p}{\dot{W}} \tag{48}$$

So, we finally obtain

$$\dot{W} = \overset{E}{\dot{W}} + \overset{S}{\dot{W}} + \underbrace{\overset{D_p}{\dot{W}} + \overset{D_v}{\dot{W}}}_{\overset{D}{\dot{W}}} \tag{49}$$

We have assumed that the changes of the internal state of the material can be regarded as a sequence of equilibrium states. Then, the specific energy is well defined in each state of the process and we may take the usual overall statement concerning the specific free energy. In so doing, however, we must be aware of the fact that into the part $\overset{S}{W}$ of the plastic work rate $\overset{P}{W}$ only the athermal stress $\bar{\tau}_C^A$ enter, since only these stresses are involved in the plastic mechanism. For the same reason, we can introduce only the athermal stresses $\bar{\tau}_C^A$ into the statement concerning the dissipated plastic work $\overset{D_p}{W}$. On the other hand, we have to add the dissipated viscous work $\overset{D_v}{W}$ to $\overset{D_p}{W}$ in order to obtain the total rate of dissipation. The different mechanisms for determining the total dissipation and their coupling have been discussed by Perzyna.[46]

We now consider an example in which the specific free energy has the following form:

$$\overset{E}{\phi} = \overset{E}{\phi}(f_C^A, T) + \overset{S}{\phi}(T, k, \alpha_C^A) = \overset{E}{\phi}(f_C^A, T) + k + f(T) + k\alpha_C^A\alpha_A^C \tag{50}$$

In this equation, h denotes a constant with the dimension of a specific energy like the variable k and the function $f(T)$.

Furthermore, we assume that the dissipation is given by

$$\overset{D_p}{W} = \frac{1}{\rho_0}\xi(\bar{\tau}_C^A - c\rho_0 h\alpha_C^A)\overset{P}{d}{}_A^C$$

$$\overset{D_v}{W} = \frac{1}{\rho_0}(\tau_C^A - \bar{\tau}_C^A)\overset{P}{d}{}_A^C \tag{51}$$

where $\xi < 1$ and c denotes constant numbers. This leads to

$$\overset{D}{W} = \overset{D_p}{W} + \overset{D_v}{W} = (\xi - 1)\overset{P}{W} - \xi ch\alpha_C^A\overset{P}{d}{}_A^C + \overset{I}{W} \tag{52}$$

Hence, we obtain

$$\overset{S}{W} = \overset{I}{W} - \overset{D}{W} = (1-\xi)\overset{P}{W} + \xi ch\alpha_C^A\overset{P}{d}{}_A^C \tag{53}$$

On the other hand, from Eqs. (42) and (50) we have

$$\overset{S}{W} = \dot{k} + 2h\alpha_C^A\overset{\nabla}{\alpha}{}_A^C \tag{54}$$

Equations (53) and (54) are compatible, for instance, if we put

$$\dot{k} = (1-\xi)\overset{P}{W} \tag{55}$$

and

$$\overset{\nabla}{\alpha}{}_A^C = \tfrac{1}{2}c\xi\overset{P}{d}{}_A^C \tag{56}$$

From Eq. (55) it follows that, in our case, the plastic work $\overset{P}{W}$ is equivalent to the thermodynamic state variable k. This is still true if we take ξ as a function of k. But it does not hold in the general case when ξ also depends on the other state variables T and α_C^A. Equation (56) shows that only in a very special case, a very unrealistic one, the state variable α_C^A is equivalent to the plastic deformation.

From the thermodynamical considerations, it follows then that we may introduce the quantities k and α_C^A, defined by Eqs. (55) and (56) or any other equivalent set $(\overset{P}{W}, c\rho_0 h\alpha_C^A)$, as internal variables into the corresponding constitutive equations of the process description.

The constitutive equations themselves are not yet determined completely by Eqs. (50), (51), (55), and (56). These given only the restrictive frame for the formulation of these equations. We may derive a complete set of constitutive equations, which is compatible with this frame, by the further assumptions:

1) The introduction of a yield condition of the form,

$$F = (\bar{t}_C^A - c\rho_0 h\alpha_C^A)(\bar{t}_C^A - c\rho_0 h\alpha_C^A) - g^2(\overset{P}{W}, T) = 0 \tag{57}$$

where $\bar{t}_C^A$ denotes the deviator of the Kirchhoff stresses $\bar{\tau}_C^A$.

2) The plastic deformation obeys the so-called normality rule,

$$\overset{P}{d}{}_C^A = \dot{\lambda}\frac{dF}{d\bar{\tau}_C^A} \tag{58}$$

3) The relations between the viscous stresses and the inelastic deformation rate are of the form,

$$\overset{P}{d}{}_C^A = \frac{1}{2\eta}\overset{*}{t}{}_C^A = \frac{1}{2\eta}(t_C^A - \bar{t}_C^A) \tag{59}$$

4) The quantities ξ and c are constant.

We can eliminate the athermal stresses $\bar{\tau}_C^A$ (which are not state variables) from the equations of evolution by considering that the inelastic deformation can be expressed in two different ways. In one, the plastic mechanism is considered and the viscous mechanism in the second. From Eq. (57), we then obtain

$$\overset{P}{d}{}_C^A = 2\dot{\lambda}(\bar{t}_C^A - c\rho_0 h\alpha_C^A) \tag{60}$$

while from Eq. (59), we have

$$\overset{P}{d}{}_C^A = \frac{1}{2\eta}(t_C^A - \bar{t}_C^A)$$

$$= \frac{1}{2\eta}\{t_C^A - c\rho_0 h\alpha_C^A - (\bar{t}_C^A - c\rho_0 h\alpha_C^A)\} \tag{61}$$

By comparing these equations for $\overset{P}{d}{}_C^A$, we get

$$\dot{\lambda} = \frac{1}{4\eta}\left\{\left(\frac{(t_C^A - c\rho_0 h\alpha_C^A)(t_C^A - c\rho_0 h\alpha_C^A)}{g^2}\right)^{1/2} - 1\right\} \tag{62}$$

Following the course of the process in each state, the internal parameters $\overset{P}{W}$ and α_C^A and, therefore, also $k^2 = k^2(\overset{P}{W}, \alpha_C^A)$ are known. Thus, we may calculate $\dot{\lambda}$ from Eq. (62) and then all the other needed quantities such as $\bar{t}_C^A$ and $\overset{P}{d}{}_C^A$.

Discussion

Many thermodynamic considerations of nonisothermal, elastic-viscoplastic deformations refer essentially to the general fundamentals that must be observed in describing such phenomena as thermomechanical processes and then discuss what particular restrictions follow from the second law of thermodynamics. Only a few papers attempt to describe completely such processes by state equations. Most of these papers introduce plastic strains as thermodynamic state variables. But one may conclude from the consideration of the phenomena in the crystal lattice (dislocations, for example, that have completely passed through the crystal produce plastic strains but no changes of state) as well as from phenomenological observations (different states of hardening

can belong to the same plastic strains) that plastic strains in general cannot be regarded as state variables. Furthermore, all these papers consider the plastic work as completely dissipated. However, this is in contradiction with experimental results, from which it emerges that one part of plastic work is used for producing states of residual stresses in the lattice, which, when phenomenologically considered, cause hardening.

The results in work presented here can be extended to more complex constitutive equations by introducing more internal parameters or state variables. We may extend our approach to more general, anisotropic hardening materials by assuming [see Eq. (50)], for example, that

$$\overset{E}{\phi}=\overset{E}{\phi}(f^A_C,T)+\overset{S}{\phi}(T,k,\alpha^A_C,A^{AB}_{CD})$$

$$=\overset{E}{\phi}(f^A_C,T)+k+f(T)+A^{AB}_{CD}\alpha^C_A\alpha^D_B \tag{63}$$

Also, it may be more advantageous to replace the assumption in Eq. (58) for the plastic deformation rate by

$$\overset{P}{d}{}^A_C=\dot{\lambda}\frac{\partial F}{\partial \tau^A_C}+B^{AB}_{CD}\overset{\nabla}{\tau}{}^D_B \tag{64}$$

This form of this model appears to be more suitable for representing some experimental results in which second-order effects and some deviations from the normality rule have been observed. Sometimes, the normality rule is considered as a fundamental law based on an entropy production principle. But we should keep in mind that, since not all of the plastic work is dissipated, we cannot expect the total plastic deformation rate to obey the theory of plastic potential even though the mentioned principles of entropy production are correct.

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